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INVISCID MODES OF INSTABILITY IN SPIRAL FLOW

by

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Inviscid modes of instability in spiral flow*

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Abstract

The stability of an inviscid fluid between rotating cylinders with an axial flow is considered under the small gap approximation. When the axial flow is small compared to the rotational velocity, a new perturbation method differing in several respects from the conventional method has been devised. It is found that the presence of the axial flow has a stabilizing effect, and that the correction to the growth rates of the pure rotation case is of second-order in the axial to rotational velocity ratio. Furthermore, the instabilities are confined to only a finite range of wave numbers.

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1. Introduction.

The effect of an axial pressure gradient on the stability of viscous flow between rotating cylinders has recently been studied both theoretically by Chandrasekhar [1-3] and Di Prima [4] and experimentally by Donnelly & Fultz [5] and Snyder [6]. This problem is of particular interest because of the interaction it exhibits between the Tollmien-Schlichting mechanism of instability associated with the axial flow and the Taylor-Görtler mechanism associated with the rotational flow. As in the usual stability theory for parallel flows in which the inviscid form of the Orr-Sommerfeld equation plays a central role, the corresponding inviscid analysis for the present problem may be expected to be of comparable importance.

In the present discussion we will make the usual small-gap approximation, i.e. we assume that the spacing between the cylinders is small compared to their mean radius. We will also restrict the discussion to velocity distributions that are physically realizable in a viscous fluid so that the angular and axial velocity distributions are given by

$$\Omega(r) \cong \Omega_1[1-(1-\mu)\zeta] \tag{1}$$

and

$$W(r) \cong 6W_m\zeta(1-\zeta), \tag{2}$$

where $\zeta = (r-R_1)/d$, $\mu = \Omega_2/\Omega_1$, and W_m is the mean velocity of the axial flow.

For an inviscid fluid, the stability of a purely rotational flow is determined by Rayleigh's circulation theorem which states that a necessary and sufficient condition for stability with respect to rotationally symmetric disturbances is that the square of the circulation should increase outwards from the axis of rotation. On this criterion, the distribution of angular velocity given by equation (1) is unstable if $\mu < 1$ and stable otherwise. On the other hand, the stability of a purely axial flow is governed by Rayleigh's inflection point theorem which states that a necessary condition for instability is that the velocity profile have an inflection point somewhere in the flow region. For the symmetrical velocity distribution (2), this condition is also sufficient. We would expect, therefore, that the stability or instability of a basic spiral flow would be determined solely by its rotational component as was first suggested by Chandrasekhar [7].

Since the axial flow is stable in the absence of rotation, we might expect it to have a stabilizing effect on an unstable rotational flow in the sense that the growth rates of the disturbances, if positive, would be decreased when an axial flow is also present and that the instability would then extend only to a finite range of wave-numbers. This conjecture is borne out by the present calculations.

When the axial flow is dominant compared to the rotational flow, however, the situation is more complicated for the governing equation then possesses a singularity in

the region of flow. In the total absence of rotation, the roots of the indicial equation relative to this singular point differ by an integer and, consequently, one of the solutions (commonly denoted by φ_2) has a logarithmic singularity. When rotation is present, however, the roots of the indicial equation no longer differ by an integer and the singular point is than a regular one. Thus, even a small amount of rotation must result in a large modification of the solution φ_2 in the neighborhood of the critical layer. This aspect of the problem will be considered further in a later paper.

In the existing non-dimensional formulations of the full viscous problem two parameters appear: the axial Reynolds number

$$R = W_m d/\nu \tag{3}$$

and the usual small-gap Taylor number

$$T = -\frac{4A\Omega_1}{\nu^2} d^4 \cong 2(1-\mu)\left(\frac{\Omega_1 R_1 d}{\nu}\right)^2 \frac{d}{R_1} . \tag{4}$$

In considering the approximate solution of this problem by asymptotic methods it is convenient to replace either R or T by a new parameter that is independent of viscosity. Thus, we define the quantity

$$\beta \equiv R/\sqrt{T}. \tag{5}$$

When the effects of rotation are dominant, the relevant parameters are clearly T and β; conversely, when the axial flow is dominant we would use R and $1/\beta$. The related inviscid problems are then obtained by formally allowing T or R to

become infinite for fixed values of β. Both of these limiting procedure are, of course, singular since the order of the governing equations is thereby reduced from six to two.

In all of the existing work on the viscous problem it has been assumed that the axial Reynolds number is small so that the effects of rotation are dominant. The present inviscid analysis will also be confined to this case where $\beta << 1$. Even in this very restricted form the problem presents a number of difficulties and it has been necessary therefore to resort to a perturbation method.

2. The governing equations.

If the steady basic flow given by equations (1) and (2) is subjected to a rotationally symmetric disturbance whose t and z dependence is of the form

$$\exp(pt+ikz), \tag{6}$$

then the linearized equation for u, the radial component of the disturbance velocity, can be written in the non-dimensional form (cf. [2])

$$(\sigma+ia\beta w)^2(D^2-a^2)u-ia(\sigma+ia\beta w)(D^2w)u = -a^2\omega u, \tag{7}$$

where

$$D = d/d\zeta, \quad a = kd, \quad \text{and} \quad \sigma = p/(-4A\Omega_1)^{\frac{1}{2}}. \tag{8}$$

In equation (7), ω and w are the non-dimensional forms of the angular and axial velocity distributions:

$$\omega(\zeta) = 1-(1-\mu)\zeta \quad \text{and} \quad w(\zeta) = \zeta(1-\zeta). \tag{9}$$

The parameter β, defined by equation (5), can also be written in the form

$$\beta = 6(W_m/\Omega_1 d)(-4A/\Omega_1)^{-\frac{1}{2}} \tag{10}$$

in which there is no longer any explicit reference to the viscous problem. The boundary conditions that must be satisfied by u are

$$u=0 \quad \text{at } \zeta=0 \text{ and } 1. \tag{11}$$

The purely rotational case ($\beta=0$) has been considered by Reid [8] who obtained an exact solution for $-\infty < \mu < 1$ in terms of Airy functions.

If we confine our attention to the case in which the cylinders rotate in the same direction, i.e. to the range $0 \le \mu \le 1$, then $\omega(\zeta)$ is of one sign and, to a good approximation, can be replaced by its average value $\omega_m = \frac{1}{2}(1+\mu)$. More precisely, if we write

$$\omega(\zeta) = \omega_m[1-\varepsilon(\zeta-\tfrac{1}{2})], \quad \text{where} \quad \varepsilon = 2\,\frac{1-\mu}{1+\mu}\,, \tag{12}$$

then it has been found, in both the viscous [2] and the inviscid [5] problems for the purely rotational case, that the correction to the eigenvalues is only of second-order in ε. This approximation is, of course, no longer valid when $\mu < 0$, since $\omega(\zeta)$ will then change sign in $0 \le \zeta \le 1$. The simplification that results from this approximation can be readily seen if we note that, in a perturbation method based on the smallness of the parameter β, the lowest approximation will be the purely rotational case with ω a constant, the solutions of which will be simply sines and cosines rather than the more complicated Airy functions. Furthermore, the analysis to be described can, in fact, be carried through for the general case with but slight modifications, and it is found that most of the important features of the problem are already exhibited in the simplified case.

Thus, with $\omega(\zeta)$ replaced by its average value ω_m, we have the governing equation in the final form

$$(\sigma+ia\beta w)^2(D^2-a^2)u-ia(\sigma+ia\beta w)(D^2w)u = -a^2\omega_m u. \tag{13}$$

For $\beta \ll 1$, an approximate solution of this equation can be obtained by the perturbation method described in the following section.

3. The perturbation method.

The method to be described here differs from the usual perturbation methods, as described, for example, by Courant and Hilbert [9], in a number of important respects the most important of which is the elimination of the crucial requirement that the normal modes of the unperturbed problem be complete. In the present unperturbed problem, for example, when the cylinders rotate in opposite directions (i.e. $\mu < 0$) the completeness question has not been satisfactorily answered. In addition to the discrete spectrum of positive and negative eigenvalues (with limit points at $\pm\infty$) that has been found, the problem may also possess a continuous spectrum (though this later possibility has not yet been investigated).

To avoid this difficulty, we shall fix our attention on a particular mode and show that it is possible to obtain corrections to the eigenvalue and eigenfunction of that mode, to arbitary orders in β, without any knowledge of the other modes. From the point of view of stability theory this method provides all of the information that is usually required. When the cylinders rotate in the same direction and ω is replaced by its average value as in equation (13), the eigenfunctions of the unperturbed problem are clearly complete and there would be no difficulty in applying one of the usual perturbation methods. Even in this case, however, the present method has the important advantage that the corrections to the eigenvalue and eigenfunction of a given mode can be obtained in terms of finite sums rather than infinite series.

The success of the present method depends crucially on being able to solve explicitly the inhomogeneous equations that result from the expansion. This has, however, proved to be possible not only in the present problem (for arbitrary rates of rotation) but also in Bisshopp's work [10] on non-rotationally symmetric inviscid modes in Couette flow. In fact, Bisshopp's work provides an example in which the "averaging" approximation was not made and the analysis was carried through in terms of Airy functions.

Before we consider the expansion for the solution u, it is convenient to introduce the transformation

$$x = x_1\zeta, \quad \text{where} \quad x_1 = \frac{a}{\sigma}(\omega_m - \sigma^2)^{\frac{1}{2}}. \tag{14}$$

As a result of this transformation the value of x corresponding to $\zeta=1$ at which one of the boundary conditions must be applied will depend on β. The function u is now of the form $u=u(x;a,\mu,\sigma,\beta)$ with $\sigma=\sigma(a,\mu,\beta)$ but their dependence on a (supposed fixed) and μ (which occurs only through ω_m) need not be indicated explicitly in what follows. We now expand u in the form

$$u = u_o(x,\sigma) + \beta u_1(x,\sigma) + \beta^2 u_2(x,\sigma) + \ldots \tag{15}$$

but do not expand σ until a later stage in the analysis. On substituting the expansion (15) into equation (13), and collecting terms of the same order in β, we obtain

$$(D^2+1)u_o = 0, \tag{16}$$

$$(D^2+1)u_1 = u_o \sum_{i=o}^{2} c_i x^i, \tag{17}$$

$$(D^2+1)u_2 = u_1 \sum_{i=o}^{2} c_i x^i + u_o \sum_{i=o}^{3} d_i x^i, \tag{18}$$

etc., where $D \equiv d/dx$, $c_i = c_i(a,\sigma,x_1)$, and $d_i = d_i(a,\sigma,x_1)$. To avoid going into too much detail, we will not give the full expressions for c_i, d_i and the other coefficients that will appear later; it suffices to indicate the parameters on which they depend. The boundary conditions now require that $u_o+\beta u_1+\beta^2 u_2+\ldots = 0$ at $x=0$ and $x=x_1$, where $x_1=x_1(\sigma)$ and σ, in turn, depends on β. Thus, in contrast to the usual perturbation methods, the position of the outer boundary is not fixed and we cannot separate the boundary conditions to obtain independent conditions for each $u_i(x)$. Instead we first construct two linearly independent solutions of equation (13) correct to $O(\beta^2)$, and the subsequent expansion of σ then allows the boundary condition at x_1 to be satisfied to $O(\beta^2)$.

Two linearly independent solutions $u_{oi}(i=1,2)$ of equation (16) are simply sin x and cos x. Due to the linearity of equations (17) and (18), particular solutions* of the inhomogeneous equations can be generated from the solutions of

$$(D^2+1)f_n = x^n u_o \tag{19}$$

*The solutions of the homogeneous equation for u_1 and u_2 need not be considered.

and

$$(D^2+1)g_n = x^n u_o', \tag{20}$$

which are, as can be easily verified,

$$f_n = \frac{1}{2n+1}\left[x^n u_o' - \frac{n}{2}x^{n-1}u_o + \frac{n(n-1)(n-2)}{2}f_{n-3}\right] \tag{21}$$

and

$$g_n = \frac{1}{2(n+1)}\left[x^{n+1}u_o - \frac{n(n+1)}{2n-1}x^{n-1}u_o' + \frac{(n+1)n(n-1)(n-2)}{2n-1}g_{n-3}\right]. \tag{22}$$

In the above formulas n is a non-negative integer, f_n and g_n with negative subscripts are to be regarded as zero identically, and f_1 is to be taken as $xu_o'/3$ instead of $(xu_o'/3 - u_o/6)$ since $(D^2+1)u_o = 0$.

Thus, corresponding to $u_o=u_{oi}$, we have, for a particular solution of equation (17),

$$u_{1i} = \sum_{n=o}^{2} C_n f_{ni}, \tag{23}$$

where f_{ni} is the solution (21) of equation (19) when u_o is u_{oi}. Collecting terms we obtain

$$u_{1i} = P_1(x)u_{oi} + P_2(x)u_{oi}', \tag{24}$$

in which

$$P_1(x) = \sum_{j=1}^{2} p_{1j}x^j, \quad P_2(x) = \sum_{j=1}^{3} p_{2j}x^j, \tag{25}$$

and the $p_{ij}=p_{ij}(a,\sigma,x_1)$ are linear combinations of the C_i's. Equation (18) then becomes

$$(D^2+1)u_2 = u_o \sum_{j=1}^{4} m_j x^j + u_o' \sum_{j=1}^{5} n_j x^j, \tag{26}$$

where m_j and n_j are combinations of c_i, d_i, and p_{ij}. Its particular solutions must therefore be of the form

$$\begin{aligned} u_{2i} &= \sum_{j=1}^{4} m_j f_{ji} + \sum_{j=1}^{5} n_j g_{ji} \\ &= Q_1(x)u_{oi} + Q_2(x)u_{oi}', \end{aligned} \tag{27}$$

in which

$$Q_1(x) = \sum_{j=1}^{6} q_{1j} x^j \ , \quad Q_2(x) = \sum_{j=1}^{5} q_{2j} x^j, \tag{28}$$

and the $q_{ij} = q_{ij}(a, \sigma, x_1)$ are linear combinations of m_i and n_i.

We have thus obtained two fundamental solutions of equation (13) which, correct to $O(\beta^2)$, can be written in the form

$$u_i(x) = u_{oi} + \beta(P_1 u_{oi} + P_2 u_{oi}') + \beta^2(Q_1 u_{oi} + Q_2 u_{oi}') + \ldots, \tag{29}$$

in which $u_{o1}(x) = \sin x$ and $u_{o2}(x) = \cos x$. Since $P_1(0) = Q_1(0) = 0$ and $u_{o1}(0) = 0$, the characteristic equation is simply

$$u_1(x_1) = 0, \tag{30}$$

and the required eigenfunction is a constant multiple of

$$\begin{aligned} u_1(x) = \sin x &+ \beta\{P_1(x)\sin x + P_2(x)\cos x\} \\ &+ \beta^2\{Q_1(x)\sin x + Q_2(x)\cos x\} + \ldots . \end{aligned} \tag{31}$$

We may remark here that in the general case where $\omega(\zeta)$ is not approximated by its average value, a different transformation $x=x(\zeta;a,\mu,\sigma)$ must be used with the result that both boundary positions are then functions of $\sigma(\beta)$. The characteristic equation will also involve both of the boundaries, x_o and x_1 say, and the required eigenfunction will then be a certain linear combination of the corresponding fundamental solutions $u_1(x)$ and $u_2(x)$, the leading terms of which will be the Airy functions $Ai(x)$ and $Bi(x)$ respectively.

It is at this stage that we will expand the eigenvalue σ in powers of β, and thereby find corrections to the growth rates of the unperturbed problem. Thus, we let

$$\sigma(\beta) = \sigma_o + \beta\sigma_1 + \beta^2\sigma_2 + O(\beta^2), \tag{32}$$

and we can then obtain the corresponding expansion for the boundary position $x_1(\sigma)$, in the form

$$\begin{aligned} x_1(\sigma) &= X_1 + \beta x_1'(\sigma_o)\sigma_1 \\ &\quad + \beta^2\{x_1'(\sigma_o)\sigma_2 + \tfrac{1}{2} x_1''(\sigma_o)\sigma_1^2\} + O(\beta^3), \end{aligned} \tag{33}$$

where $X_1 = x_1(\sigma_o)$, $x_1'(\sigma_o) = (dx_1/d\sigma)_{\beta=o}$, and $x_1''(\sigma_o) = (d^2x_1/d\sigma^2)_{\beta=o}$. In a similar manner we obtain,

$$\begin{aligned} \sin x_1 &= \sin X_1 + \beta x_1'(\sigma_o)\sigma_1 \cos X_1 \\ &+ \beta^2\{[x_1'(\sigma_o)\sigma_2 + \tfrac{1}{2} x_1''(\sigma_o)\sigma_1^2]\cos X_1 - \tfrac{1}{2}[x_1'(\sigma_o)\sigma_1]^2 \sin X_1\} \\ &+ O(\beta^3), \end{aligned}$$

$$\left.\begin{aligned}
\cos x_1 &= \cos X_1 - \beta x_1'(\sigma_o)\sigma_1 \beta \sin X_1 + O(\beta^2),\\
P_1(x_1) &= \bar{P}_1(X_1) + \beta\left[\frac{dP_1(x_1)}{d\sigma}\right]_{\beta=o}\sigma_1 + O(\beta^2),\\
&\text{and}\\
Q_1(x_1) &= \bar{Q}_1(X_1) + O(\beta),
\end{aligned}\right\} \quad (34)$$

where $\bar{P}_1(X_1) = [P_1(x_1)]_{\beta=o}$ and $\bar{Q}_1(X_1) = [Q_1(x_1)]_{\beta=o}$. On substituting these results into the characteristic equation (30) and collecting terms of the same order in β, we obtain

$$\sin X_1 = 0, \quad (35)$$

$$\sigma_1 = -\bar{P}_2(X_1)/x_1'(\sigma_o), \quad (36)$$

and

$$\sigma_2 = -\frac{1}{x_1'(\sigma_o)}\left\{\bar{Q}_2(X_1) + \left[\frac{dP_2(x_1)}{d\sigma}\right]_o \sigma_1 + \bar{P}_1(X_1)x_1'(\sigma_o)\sigma_1 + \frac{1}{2}x_1''(\sigma_o)\sigma_1^2\right\}. \quad (37)$$

Equation (35) requires that $X_1 = \pm n\pi$ $(n=1,2,\ldots)$ and from equation (14) we then obtain

$$\sigma_o = \pm\lambda\left(\frac{\omega_m}{1+\lambda^2}\right)^{\frac{1}{2}}, \quad (38)$$

where

$$\lambda = a/|X_1| = a/n\pi . \quad (39)$$

Thus, in the absence of an axial flow, there exist two modes for each value of n, one being stable and the other unstable; these modes correspond to the so-called "convective modes" in

the viscous treatment of the problem. By using the expressions for p_{1j}, $x_1'(\sigma_o)$, and σ_o, we obtain the first-order correction to the growth rate in the form

$$\sigma_1 = \frac{ia}{6}\left\{\frac{3}{x_1^2}\frac{(1-\lambda^2)}{(1+\lambda^2)} - 1\right\}. \qquad (40)$$

Since σ_1 is purely imaginary, the correction to the growth rate is of second-order in β. Only one value of σ_1 obtains for each n, and it is independent of the rotation rate μ since ω_m does not appear in equation (40). The required expression for σ_2 can conveniently be written in the form

$$x_1^2\,\frac{(1+\lambda^2)^2}{\lambda^2}\,\sigma_o\sigma_2 = -\,[A(1+\lambda^2)^3 + B(1+\lambda^2)^2 + \tfrac{1}{2}] \qquad (41)$$

where $A = -\frac{7}{4} + \frac{5x_1^2}{24} - \frac{x_1^4}{360}$ and $B = \frac{15}{8} - \frac{x_1^2}{4} + \frac{x_1^4}{120}$. (42)

Thus, for each value of n, there are also two values of σ_2, the signs of which depend on those of σ_o; σ_2 also depends on μ, but this dependence is only through σ_o.

The form given by equation (31) for the eigenfunction is not entirely satisfactory since the parameter β occurs not only in the coefficients $p_{1j}(a,\sigma,x_1)$, $q_{1j}(a,\sigma,x_1)$, but also in the variable $x=x_1\zeta$. We would, of course, prefer to have u_1 expressed in the form

$$u_1 = u_1^{(o)} + \beta u_1^{(1)} + \beta^2 u_1^{(2)} + O(\beta^3), \qquad (43)$$

in which the $u_1^{(i)}$ are independent of β. This can, of course,

be achieved by employing the same expansion procedures as those described above in obtaining σ_1 and σ_2. The details of this calculation are somewhat lengthy and in the following section, therefore, we will present only the results for a particular wave number, namely $a = \pi$.

4. Results for the first unstable mode.

We are mainly interested in the first unstable mode, for which $X_1 = +\pi$. For this mode, the dependence of σ_o, σ_1 and σ_2 on the wave-number a is shown in Fig. 1. We note that σ_2 is negative for all wave-numbers, so that the axial flow does have a stabilizing effect as expected. Over most of the range of wave-numbers, the values of σ_o, σ_1 and σ_2 agree very closely with their asymptotic behaviors:

$$\left.\begin{aligned} \sigma_o &= \frac{\sqrt{\omega_m}}{\pi}\, a\left[1 - \frac{a^2}{2\pi^2} + O(a^4)\right], \text{ as } a \to 0, \\ &= \sqrt{\omega_m}\left[1 - \frac{\pi^2}{2a^2} + O(a^{-4})\right], \text{ as } a \to \infty, \end{aligned}\right\} \quad (44)$$

$$\left.\begin{aligned} \sigma_1 &= \frac{ia}{6}\left[\left(1 - \frac{3}{\pi^2}\right) + \frac{6a^2}{\pi^4} + O(a^4)\right], \text{ as } a \to 0, \\ &= -\frac{ia}{6}\left[\left(1 + \frac{3}{\pi^2}\right) - \frac{6}{a^2} + O(a^{-4})\right], \text{ as } a \to \infty, \end{aligned}\right\} \quad (45)$$

and

$$\left.\begin{aligned} \sigma_2 &= -\frac{\pi(A_1 + B_1 + \frac{1}{2})}{\sqrt{\omega_m}}\, a\left[1 + \left(\frac{3A_1 + 2B_1}{A_1 + B_1 + \frac{1}{2}} - \frac{3}{2}\right)\frac{a^2}{\pi^2} + O(a^4)\right], \\ &\qquad \text{as } a \to 0, \\ &= -\frac{A_1}{\pi^6\sqrt{\omega_m}}\, a^4\left[1 + \left(\frac{3A_1 + B_1}{A_1} - \frac{3}{2}\right)\frac{\pi^2}{a^2} + O(a^{-4})\right], \\ &\qquad \text{as } a \to \infty, \end{aligned}\right\} \quad (46)$$

where A_1 and B_1 are the values of A and B when $X_1 = \pi$, namely

$$A_1 = 0.03558677 \text{ and } B_1 = 0.2193413. \quad (47)$$

For this mode, we have also obtained the corrections to the unperturbed eigenfunction, $u_1^{(1)}$ and $u_1^{(2)}$, for one particular wave-number*, $a = \pi$. These results can be written in the form

$$-i\sqrt{\omega_m}\, u_1^{(1)} = \sqrt{2}\pi \left[\sin \pi\zeta \sum_{j=1}^{2} K_{1j}\zeta^j + \cos \pi\zeta \sum_{j=1}^{3} K_{2j}\zeta^j\right], \quad (48)$$

in which

$$\left.\begin{aligned} &K_{11} = 1 \;, \quad K_{12} = -1 \;, \\ &K_{21} = \pi/3 \;, \quad K_{22} = -\pi \;, \quad K_{23} = 2\pi/3 \;; \end{aligned}\right\} \quad (49)$$

and

$$\omega_m u_1^{(2)} = \sin \pi\zeta \sum_{j=1}^{6} R_{1j}\zeta^j + \cos \pi\zeta \sum_{j=1}^{5} R_{2j}\zeta^j \;, \quad (50)$$

in which

$$\left.\begin{aligned} &R_{11} = \left(\frac{\pi^2}{3} - 6\right) \;, \quad R_{12} = \pi^2\left(\frac{\pi^2}{9} - \frac{7}{3} + \frac{6}{\pi^2}\right), \quad R_{13} = -\pi^2\left(\frac{2\pi^2}{3} - 4\right), \\ &R_{14} = \pi^2\left(\frac{13\pi^2}{9} - 2\right), \quad R_{15} = -\frac{4\pi^4}{3} \;, \quad R_{16} = \frac{4\pi^4}{9} \;, \\ &R_{21} = -\pi\left(\frac{2\pi^2}{45} + 2\right), \quad R_{22} = -\pi\left(\frac{\pi^2}{3} - 6\right) \;, \quad R_{23} = \pi\left(\frac{16\pi^2}{9} - 4\right) \;, \\ &R_{24} = -\frac{7\pi^3}{3} \;, \quad R_{25} = \frac{14\pi^3}{15} \;. \end{aligned}\right\} \quad (51)$$

These results are shown in Figs. 2 and 3. We note that $u_1^{(1)}$ is imaginary while $u_1^{(2)}$ is real, and that both are symmetric with respect to $\zeta = 1/2$. Indeed, this symmetry holds for any mode $X_1 = \pm n\pi$ and any wave-number a, since both the differential equation (13) and the boundary conditions are symmetric.

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* This value of a was chosen partly for convenience and partly because it closely approximates the critical value of a in the related viscous problem.

Such symmetry will of course not obtain if $\omega(\zeta)$ is not approximated by its average. [At this stage it might seem that the algebra involved would have been reduced had we carried through the analysis on the interval $(-\frac{1}{2},+\frac{1}{2})$ rather than (0,1). This is not so, however, for the characteristic equation would not then be as simple as equation (30) and the eigenfunction will be a linear combination of both u_1 and u_2, thus doubling the number of polynomials involved. Furthermore, the symmetry condition provides a useful check on the analysis.]

From Figs. 2 and 3 it may also be noticed that the functions $iu_1^{(1)}$ and $u_1^{(2)}$, if suitably normalized, are very nearly sine curves. This means that the actual deviation of the eigenfunction from that of the purely rotational case is much slighter than might at first have been expected.

5. Concluding Remarks.

In the general case in which $\omega(\zeta)$ is not approximated by its average value, two distinct types of modes may be present in the purely rotational case [8]. For $0 \leq \mu \leq 1$, only the "convective modes" ($\sigma_o^2 \geq 0$) are present, whereas for $\mu < 0$, there also exist "oscillatory modes" ($\sigma_o^2 < 0$). The same perturbation scheme can equally well be employed here, though it will be more laborious to obtain numerical results. Qualitatively it has been found that, for the convective modes, σ_1 is imaginary and has two values, one for each value of σ_o. For the oscillatory modes, σ_1 again has two values, one for each σ_o, but they are real and therefore the effect (stabilizing or otherwise) of the axial flow on these modes can be determined without going to second-order in β. The values of σ_1 will, however, depend on μ in the general case.

One of the main qualitative conclusions that can be drawn from this work is that, when an axial flow is present, there is only a finite range of wave-numbers with positive growth rates. Indeed, we can easily show that the equation $\sigma_o + \beta^2\sigma_2 = 0$ does possess a positive root, $a = a_o$ say, and hence for $a > a_o$ the flow is stable. From equations (38) and (41), it is more convenient to find first the real roots of

$$A_1\alpha^3 + B_1\alpha^2 - C_1\alpha + \frac{1}{2} = 0, \tag{52}$$

where $\alpha = 1 + a^2/\pi^2$ and $C_1 = \pi^2\omega_m/\beta^2$. It can then be shown

that only one root of equation (52) leads to real positive values of a_o^2, and from this root we obtain

$$a_o^2 = \pi^3\sqrt{\frac{\omega_m}{A_1}}\ \frac{1}{\beta}\left[1 - \frac{1}{\pi}\sqrt{\frac{A_1}{\omega_m}}\left(1 + \frac{B_1}{2A_1}\right)\beta + O(\beta^2)\right]. \quad (53)$$

Thus, the range of wave-numbers with positive growth rates will be smaller the more dominant the axial flow is relative to the rotational flow.

The differential equation (13) possesses a singularity where $\sigma + i a\beta w = 0$. Since σ is complex, this singularity does not enter the flow region. Nor will the real part of the singularity lie in $0 \leq \zeta \leq 1$ if $\beta \ll 1$. Indeed, if we write $x = \zeta - \frac{1}{2}$ and denote the position of the singularity by $x = x^* = x_1^* + i x_2^*$ (x_1^* and x_2^* real), it can be easily seen that, for wave-numbers $a \ll \beta^{-1}$, $(x_1^*)^2 = (\sigma_o/2a)\beta^{-1}\{1 + O(\beta)\}$, while for $a = O(\beta^{-1})$ or larger, $(x_1^*)^2$ is at least of the order of β^{-2}. Thus in any case $(x_1^*)^2 \gg 1/4$.

The present paper deals with only one extreme case of the problem of spiral flow. The other extreme is the case where $\beta^{-1} \ll 1$. As mentioned in the introduction, the presence of even a very slight amount of rotation will drastically alter the nature of one of the solutions (viz. φ_2) of the case $\beta^{-1} = 0$. This problem may presumably also yield to a similar perturbation method, but complexities of a higher order must be expected. The general case of arbitrary β is of course of more interest and greater importance. A simple scheme such as the one used here certainly will not apply, and a detailed study of the singularities of the differential equation must first be made.

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Table I

The values of σ_o, σ_1, and σ_2 vs. a for the first amplified mode ($X_1 = \pi$)

a/π	$\sigma_o/\sqrt{\omega_m}$	$-i\sigma_1$	$\sqrt{\omega_m}\,\sigma_2$
0.1	0.09950 372	-0.03675 954	-0.00759 0476
0.2	0.19611 61	-0.07533 730	-0.01485 087
0.3	0.28734 79	-0.11721 79	-0.02154 695
0.4	0.37139 06	-0.16333 94	-0.02759 600
0.5	0.44721 36	-0.21405 29	-0.03306 799
0.6	0.51449 58	-0.26922 14	-0.03814 681
0.7	0.57346 24	-0.32838 61	-0.04307 795
0.8	0.62469 50	-0.39092 99	-0.04812 384
0.9	0.66896 47	-0.45620 27	-0.05353 535
1	0.70710 68	-0.52359 88	-0.05953 901
2	0.89442 72	-1.23818 4	-0.18907 66
3	0.94868 33	-1.95276 9	-0.55770 69
4	0.97014 25	-2.65611 8	-1.38035 0
5	0.98058 07	-3.35255 6	-2.95862 7
6	0.98639 39	-4.04490 4	-5.68148 1
8	0.99227 79	-5.42285 4	-16.29982
10	0.99503 72	-6.79602 4	-38.81890

Table II

$u_1^{(1)}$ and $u_1^{(2)}$ vs. ζ for the first amplified mode ($X_1 = \pi$)

ζ	$-i\sqrt{\omega_m}\, u_1^{(1)}(\zeta)$	$-\omega_m u_1^{(2)}(\zeta)$
0	0	0
0.1	0.442153	0.721225
0.2	0.779177	1.26810
0.3	0.984531	1.63120
0.4	1.08311	1.84136
0.5	1.11072	1.91127

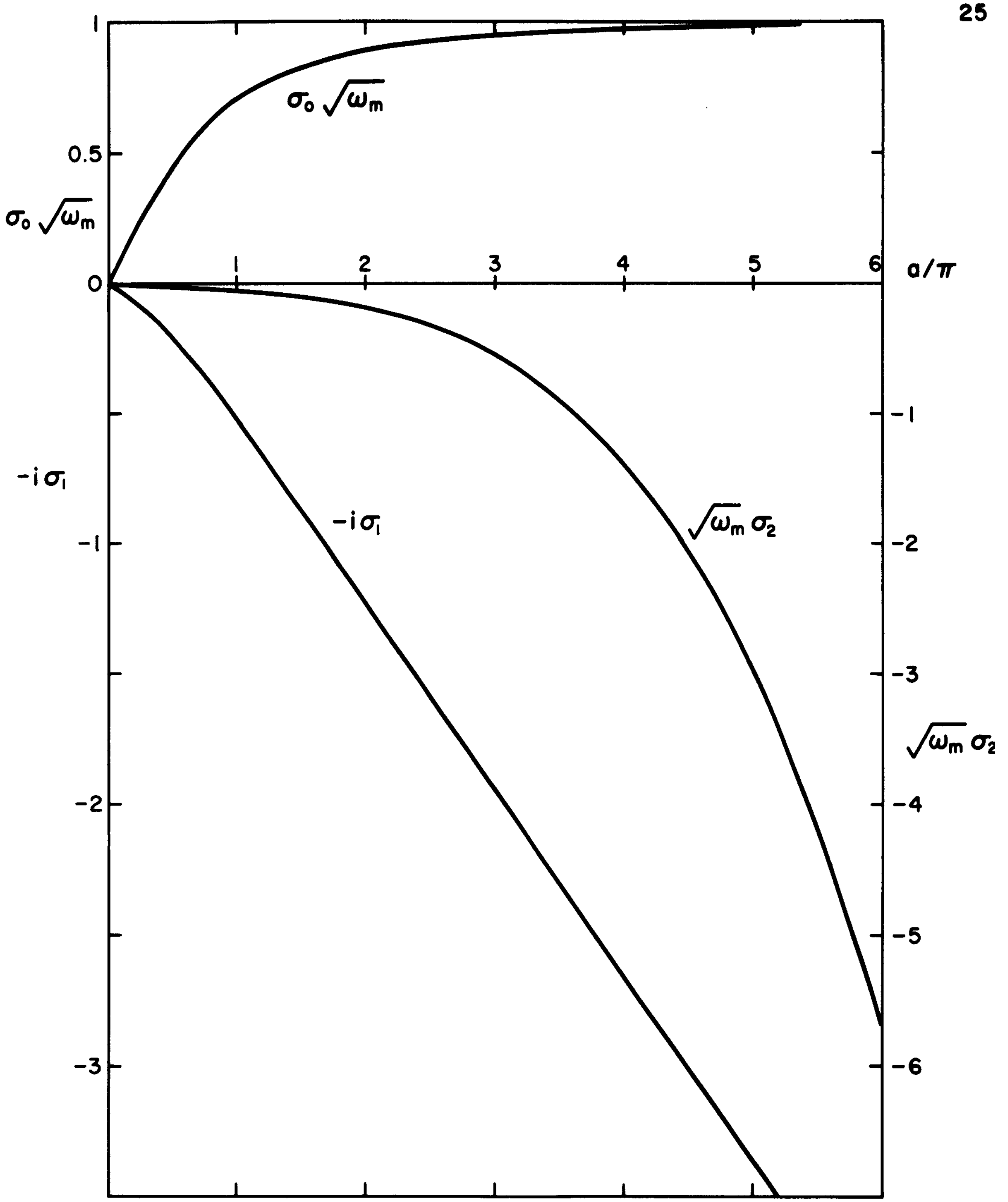

FIG. 1 THE GROWTH RATES FOR THE FIRST UNSTABLE MODE

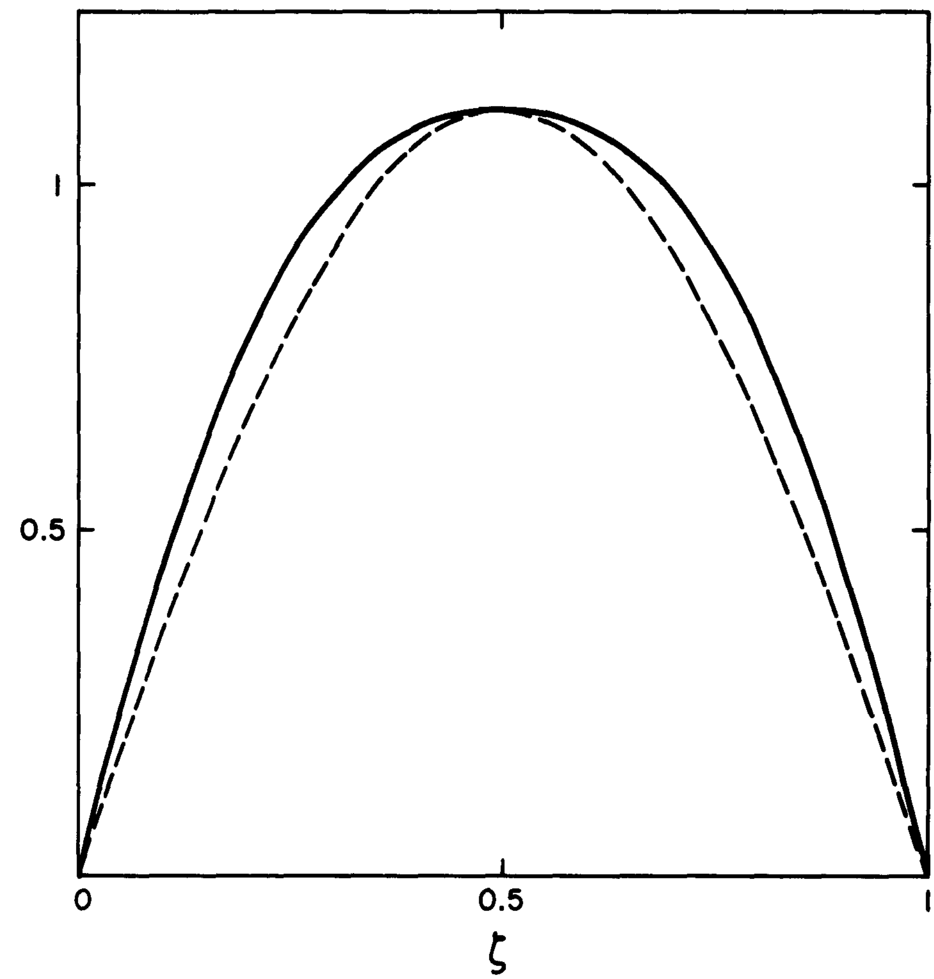

FIG. 2 $-i\sqrt{\omega_m}\,u_1^{(1)}$ (SOLID LINE) COMPARED TO 1.11072 sin $\pi\xi$ (DASHED LINE)

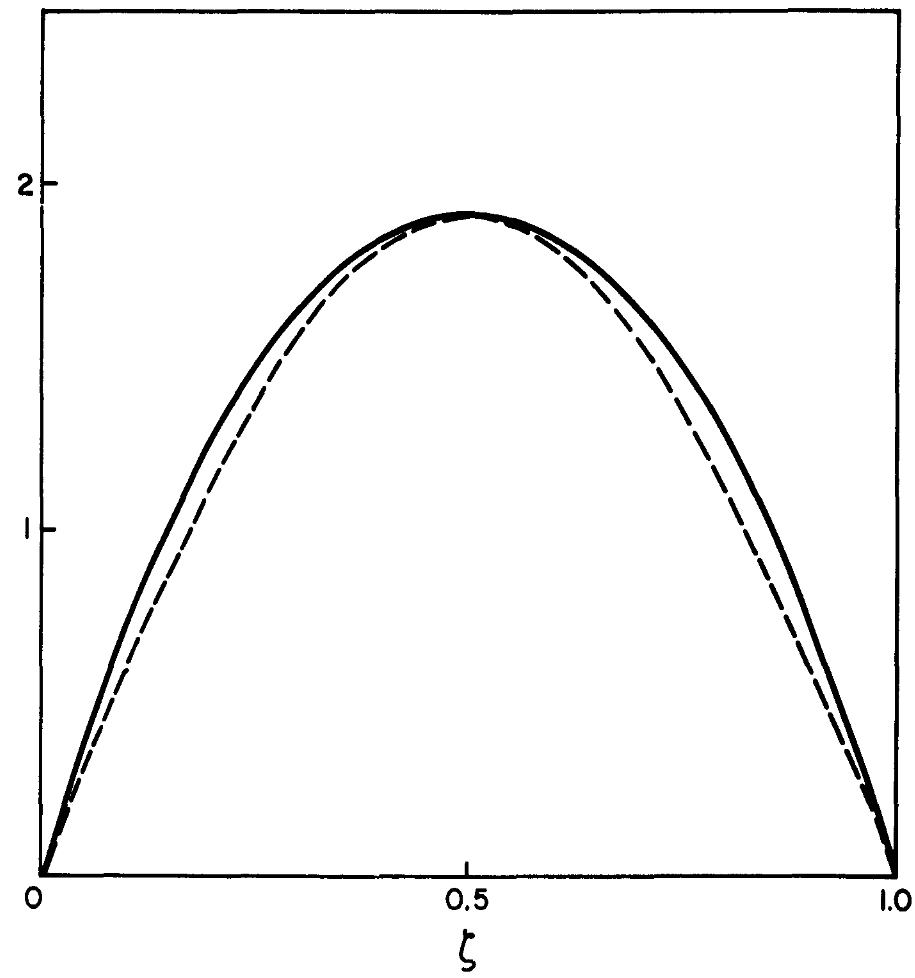

FIG. 3 $-\omega_m u_1^{(2)}$ (SOLID LINE) COMPARED TO
$1.91127 \sin \pi\xi$ (DASHED LINE)